TITLE:

Propagation, dispersion, and creation of singularities of solutions for Schrodinger equations (Spectral and Scattering Theory and Related Topics)

AUTHOR(S):

Doi, Shin-ichi

CITATION:

Doi, Shin-ichi. Propagation, dispersion, and creation of singularities of solutions for Schrodinger equations (Spectral and Scattering Theory and Related Topics). 数理解析研究所講究録 2004, 1364: 89-96

ISSUE DATE:

2004-04

URL:

http://hdl.handle.net/2433/25321

RIGHT:

京都大学学術情報リポジトリ KURENAI 紅 Kyoto University Research Information Repository

数理解析研究所講究録 1364 巻 2004 年 89-96

Propagation, dispersion, and creation of singularities of solutions for Schrödinger equations

大阪大学大学院理学研究科 土居伸一 (Shin-ichi DOI)
Department of Mathematics, Graduate School of Science, Osaka University

1. Introduction

Aim. This note is concerned with the singularities of solutions for Schrödinger equations, especially for those associated with perturbed harmonic oscillators. Our aim is to clarify how the potential of lower order, or the "subprincipal symbol," affects the singularities of solutions at resonant times.

Symbol spaces. Set $\langle z\rangle=(1+|z|^2)^{1/2}$, $z\in\mathbf{R}^n$. The symbol space $S^m(\mathbf{R}^n)$ (resp. $S_+^m(\mathbf{R}^n)$), $m\in\mathbf{R}$, is the set of all $a\in C^\infty(\mathbf{R}^n)$ such that for every $\alpha\in\mathbf{N}_0^n=(\mathbf{N}\cup\{0\})^n$

$$|\partial_z^\alpha a(z)|\le C_\alpha\langle z\rangle^{m-|\alpha|},\quad z\in\mathbf{R}^n.$$
$$\left(\text{resp. } |\partial_z^\alpha a(z)|\le C_\alpha\langle z\rangle^{\max\{m-|\alpha|,0\}},\quad z\in\mathbf{R}^n.\right)$$

Function spaces. For $s\in\mathbf{R}$, set $\langle D\rangle=(1-\Delta)^{1/2}$ and $H^s=\{f\in\mathcal{S}'(\mathbf{R}^d);\langle D\rangle^s f\in L^2(\mathbf{R}^d)\}$. The space $\mathcal{B}^s(\mathbf{R}^d)$ is the completion of $\mathcal{S}(\mathbf{R}^d)$ with respect to the norm $\|\Lambda^s\cdot\|$, where $\Lambda=(1-\Delta+|x|^2)^{1/2}$ and $\|\cdot\|=\|\cdot\|_{L^2(\mathbf{R}^d)}$. The operator Λ^s is known to have the Weyl symbol $\sigma(\Lambda^s)$ satisfying $\sigma(\Lambda^s)-(1+|x|^2+|\xi|^2)^{s/2}\in S^{s-2}(\mathbf{R}^{2d})$.

Hamilton flow. The Hamiton vector field of $f\in C^\infty(T^*\mathbf{R}^d)$ is denoted by H_f: $H_f=\sum_{j=1}^d\left(\frac{\partial f}{\partial\xi_j}\frac{\partial}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial}{\partial\xi_j}\right)$. The Hamilton flow of f is the H_f flow, which is denoted by e^{tH_f}; $(x(t,y,\eta),\xi(t,y,\eta))=e^{tH_f}(y,\eta)$ is the solution of the canonical equations

$$\dot{x}_j(t)=\partial_{\xi_j}f(x(t),\xi(t)),\qquad x_j(0)=y_j,$$
$$\dot{\xi}_j(t)=-\partial_{x_j}f(x(t),\xi(t)),\quad \xi_j(0)=\eta_j\qquad(1\le j\le d).$$

Related works. We recall some related results. Let $H=-\frac{1}{2}\Delta+V(x)$ be a Schrödinger operator on $\mathbf{R}^d$ with $V\in C^\infty(\mathbf{R}^d,\mathbf{R})$. Under some condition, $H|_{C_0^\infty(\mathbf{R}^d)}$ is essentially self-adjoint. Let H denote its closure by abuse of notation. The propagator e^{-itH}, defined first by the spectral theorem, can be extended to various continuous operators; if $V\in S_+^2(\mathbf{R}^d)$, then the mapping $\mathcal{X}\ni\phi\mapsto e^{-itH}\phi\in C(\mathbf{R},\mathcal{X})$, is continuous for $\mathcal{X}=\mathcal{B}^s(\mathbf{R}^d)$, $\mathcal{S}(\mathbf{R}^d)$, $\mathcal{S}'(\mathbf{R}^d)$. Let $K(t,x,y)$ be the distribution kernel of e^{-itH}.

(i) If $V\in S_+^2(\mathbf{R}^d)$, then there exists $T>0$ such that $K(t,x,y)$ is C^∞ in t,x,y when $0<|t|<T$ (Fujiwara [6]). If in addition $\lim_{|x|\to\infty}|\nabla^2V(x)|=0$, then $K(t,x,y)$ is C^∞

in t, x, y when $t \neq 0$ (Yajima [12]; cf. Kapitanski and Rodianski [7]). Forgetting the condition $V \in S^2_+(\mathbf{R}^d)$ now, assume $d = 1$, $V(x) \geq C(1 + |x|)^{2+\varepsilon}$ near infinity for some $\varepsilon > 0$ as well as other technical conditions. Then $K(t, x, y)$ is nowhere C^1 in t, x, y ([12]). See also [2].

(ii) Let $V(x) = \frac{1}{2}\omega^2|x|^2 + W(x)$ with $\omega > 0$ and $W \in S^2_+(\mathbf{R}^d)$ such that $|\nabla^2 W(x)| = o(1)$ as $|x| \to \infty$. Then $K(t, x, y)$ is C^∞ in t, x, y when $t \notin (\pi/\omega)\mathbf{Z}$ (Kapitanski, Rodianski and Yajima [8]). If in addition $W \in S^\lambda(\mathbf{R}^d)$ for some $\lambda < 1$, then

$$WF\, u(k\pi/\omega) = \{(-1)^k(y, \eta); (y, \eta) \in WF\, u_0\}, \quad k \in \mathbf{Z},$$

for every $u_0 \in \mathcal{S}'(\mathbf{R}^d)$ (cf. Weinstein [10], Zelditch [14]; [8]; Ōkaji [9]). In these cases, no influence of W appears. See also Wunsch [11].

(iii) Let $V(x) = \frac{1}{2}\omega^2|x|^2 + W(x)$ with $\omega > 0$ and $W \in S^\lambda_+(\mathbf{R}^d)$ for some $1 < \lambda < 2$, and assume

$$C_1\langle x\rangle^{\lambda-2} I_d \leq \nabla^2 W(x) \leq C_2\langle x\rangle^{\lambda-2} I_d$$

near infinity for some $C_1, C_2 > 0$. Then $K(k\pi/\omega, x, y)$ is C^∞ in x, y for every $k \in \mathbf{Z} \setminus \{0\}$ (Yajima [12]). In my knowledge, this is the only result that shows the influence of W.

(iv) For other results, see Craig, Kappeler, and Strauss [1] and the references therein.

2. Dispersion of Singularities (cf. [4])

In Sections 2 and 3, we consider a Schrödinger operator in the following form:

$$H = -\frac{1}{2}\Delta + \frac{1}{2}\langle Qx, x\rangle + W(x), \quad x \in \mathbf{R}^d.$$

Here Q is a $d \times d$ real symmetric matrix, and $W \in C^\infty(\mathbf{R}^d, \mathbf{R})$ satisfies $W(x) = o(|x|^2)$ as $|x| \to \infty$. Let $h_0(x, \xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}\langle Qx, x\rangle$ and $h(x, \xi) = h_0(x, \xi) + W(x)$, and set $e^{tH_h}(y, \eta) = (x(t, y, \eta), \xi(t, y, \eta))$.

In this section we consider the case where $Q = \omega^2 I$ with some $\omega > 0$ for simplicity (see [4] for further results). We consider the following conditions on W.

(W1) $W \in S^{1+\delta}_+(\mathbf{R}^d)$ for some $0 < \delta < 1$; $|\nabla^2 W(x)| = o(1)$ as $|x| \to \infty$.

(W2) There exist $F_1, \ldots, F_d \in C(\mathbf{R}^d \setminus \{0\}, \mathbf{R})$, homogeneous of degree δ (where δ is the constant in (W1)), such that with $F = (F_1, \ldots, F_d)$,

$$\lim_{|x|\to\infty} |\nabla W(x) - F(x)|/|x|^\delta = 0.$$

Under (W1) and (W2), define $\theta_k \in C(\mathbf{R}^d \setminus \{0\}, \mathbf{R}^d)$ by

$$\theta_k(\eta) = \begin{cases} C_{\delta,\omega} n(F(\eta) - F(-\eta)) & \text{if } k = 2n,\ n \in \mathbf{Z}; \\ C_{\delta,\omega}\big(n(F(\eta) - F(-\eta)) + F(\eta)\big) & \text{if } k = 2n+1,\ n \in \mathbf{Z}. \end{cases}$$

Here $C_{\delta,\omega} = \int_0^\pi |\sin t|^{1+\delta} dt/\omega^{2+\delta}$. This function θ_k is closely related to the asymptotic behavior of $e^{tH_h}(y,\eta)$ as $|\eta| \to \infty$. More precisely we have

Proposition 1. *Let $k \in \mathbf{Z} \setminus \{0\}$ and $I = [\frac{k\pi}{\omega} - \varepsilon, \frac{k\pi}{\omega} + \varepsilon]$ for $0 < \varepsilon \ll 1$. Then*

$$x(-t,y,\eta) = (-1)^k\big(\theta_{-k}(\eta) - (t - k\pi/\omega)\,\eta\big) + r_1(t,\eta) + (t - k\pi/\omega) r_2(t,\eta) + r_3(t,y,\eta)$$

for $t \in I$, $y, \eta \in \mathbf{R}^d$. Here $|r_1(t,\eta)| = o(|\eta|^\delta)$ as $|\eta| \to \infty$ uniformly in $t \in I$; $|r_2(t,\eta)| \leq C(\langle\eta\rangle^\delta + |t - k\pi/\omega|\langle\eta\rangle)$ and $r_3(t,y,\eta) \leq C\langle y\rangle$ with some $C > 0$.

The next lemmas give a sufficient condition for $\theta_k(\eta)$ to be nonzero.

Lemma 2. *Assume (W1) and (W2). Assume that for some $c_0 > 0$ and $C_0 > 0$,*

$$\nabla^2 W(x) \geq c_0 |x|^{\delta-1} I \quad \text{if } |x| \geq C_0.$$

Then $\theta_k(\eta) \neq 0$ for every $k \in \mathbf{Z} \setminus \{0\}$ and $\eta \in \mathbf{R}^d \setminus \{0\}$.

To state our microlocal dispersion theorems, we need

Definition 3. For $a \in S^0(\mathbf{R}^d)$, denote by $\operatorname{Char} a$ the set of all $\eta \in \mathbf{R}^d \setminus \{0\}$ such that $\liminf_{t\to\infty} |a(t\eta)| = 0$.

We state our microlocal dispersion theorem at resonant times.

Theorem 4. *Assume (W1) and (W2). Let $k \in \mathbf{Z}$ and $\eta_0 \in \mathbf{R}^d \setminus \{0\}$ such that $\theta_{-k}(\eta_0) \neq 0$. Let $u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ and $u(t) = e^{-itH} u_0 \in C(\mathbf{R}_t, \mathcal{B}^{s_0}(\mathbf{R}^d))$, and let $r > 0$. Assume $\langle x\rangle^r a(x) u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ for some $a \in S^0(\mathbf{R}^d)$ satisfying* $\operatorname{Char} a \not\ni (-1)^k\theta_{-k}(\eta_0)$. *Then there exists $b \in S^0(\mathbf{R}^d)$,* $\operatorname{Char} b \not\ni \eta_0$, *such that*

$$\langle x\rangle^{-r} \langle D\rangle^{\delta r} b(D) u(k\pi/\omega) \in \mathcal{B}^{s_0}(\mathbf{R}^d).$$

We consider a uniform estimate of the solution near resonant times.

Theorem 5. *Assume (W1) and (W2). Let $k \in \mathbf{Z}$ and $\eta_0 \in \mathbf{R}^d \setminus \{0\}$ such that $\theta_{-k}(\eta_0) \notin \{t\eta_0; t \geq 0\}$. Let Γ be the minimal closed, convex cone of $\mathbf{R}^d \setminus \{0\}$ containig $\theta_{-k}(\eta_0)$ and $-\eta_0$. Let $u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ and $u(t) = e^{-itH}u_0 \in C(\mathbf{R}_t, \mathcal{B}^{s_0}(\mathbf{R}^d))$, and let $r > 0$. Assume that $\langle x\rangle^r a(x)u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ for some $a \in S^0(\mathbf{R}^d)$ satisfying* $\operatorname{Char} a \cap (-1)^k\Gamma = \emptyset$. *Then there exists $b \in S^0(\mathbf{R}^d)$,* $\operatorname{Char} b \not\ni \eta_0$, *such that*

$$\langle x\rangle^{-r}\big(\langle D\rangle^\delta + |t - k\pi/\omega|\langle D\rangle\big)^r b(D)u(t) \in C([k\pi/\omega, k\pi/\omega + \varepsilon], \mathcal{B}^{s_0}(\mathbf{R}^d)).$$

Remark. The relation between the direction and order of the decay of the initial data u_0 and those of the regularity of the solution $u(t)$ is sharp. □

For comparison we state the microlocal dispersion theorem at nonresonant times.

Theorem 6. *Assume that $W \in S^2_+(\mathbf{R}^d)$ and that $|\nabla W(x)| = o(|x|)$ as $|x| \to \infty$. Let $\eta_0 \in \mathbf{R}^d \setminus \{0\}$, $k \in \mathbf{Z}$, and $r > 0$. Let $u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ and $u(t) = e^{-itH}u_0 \in C(\mathbf{R}_t, \mathcal{B}^{s_0}(\mathbf{R}^d))$. Assume that $\langle x\rangle^r a(x)u_0 \in \mathcal{B}^{s_0}(\mathbf{R}^d)$ for some $a \in S^0(\mathbf{R}^d)$ satisfying* $\operatorname{Char} a \not\ni (-1)^{k+1}\eta_0$. *Then there exists $b \in S^0(\mathbf{R}^d)$,* $\operatorname{Char} b \not\ni \eta_0$, *such that*

$$\langle x\rangle^{-r}\langle D\rangle^r b(D)u(t) \in C\big((k\pi/\omega, (k+1)\pi/\omega), \mathcal{B}^{s_0}(\mathbf{R}^d)\big).$$

3. Propagation and creation of Singularities (cf. [3, 5])

Let H be the Schrödinger operator in Section 2. In this section we consider the case where $\langle Qx, x\rangle = \sum_{j=1}^d \omega_j^2 x_j^2$ for some $\omega_1, \dots, \omega_d > 0$. Set $A(t) = \operatorname{diag}(\cos\omega_1 t, \cdots, \cos\omega_d t)$ and $B(t) = \operatorname{diag}(\frac{\sin\omega_1 t}{\omega_1}, \cdots, \frac{\sin\omega_d t}{\omega_d})$. Then we have

$$e^{tH_{h_0}}(y,\eta) = (A(t)y + B(t)\eta, A'(t)y + B'(t)\eta).$$

We consider the following conditions on W.

(W3) $W \in S^1(\mathbf{R}^d)$.

(W4) There exist $F_1, \dots, F_d \in C(\mathbf{R}^d \setminus \{0\}, \mathbf{R})$, homogeneous of degree 0, such that with $F = (F_1, \dots, F_d)$,

$$\lim_{|x|\to\infty} |\nabla W(x) - F(x)| = 0.$$

Under (W3) and (W4), define

$$\tilde{x}(t,s,y,\eta) = y + \int_s^t B(\tau)F(B(\tau)\eta)\,d\tau, \quad \tilde{\xi}(t,s,y,\eta) = \eta - \int_s^t A(\tau)F(B(\tau)\eta)\,d\tau;$$
$$\tilde{\Phi}_{ts}(y,\eta) = (\tilde{x}(t,s,y,\eta), \tilde{\xi}(t,s,y,\eta)), \quad \Phi_{ts}(y,\eta) = e^{-tH_{h_0}} \circ e^{(t-s)H_h} \circ e^{sH_{h_0}}(y,\eta).$$

Proposition 7. *For every compact set $K \subset \mathbf{R}^d$ and every compact interval $I \subset \mathbf{R}$,*

$$\lim_{|\eta|\to\infty} \sup_{s,t\in I, y\in K} \left|\Phi_{ts}(y,\eta) - \tilde{\Phi}_{ts}(y,\eta)\right| = 0.$$

In particular,

$$\lim_{|\eta|\to\infty} \sup_{t\in I, y\in K} |x(t,y,\eta) - A(t)\tilde{x}(t,0,y,\eta) - B(t)\tilde{\xi}(t,0,y,\eta)| = 0;$$

$$\lim_{|\eta|\to\infty} \sup_{t\in I, y\in K} |\xi(t,y,\eta) - A'(t)\tilde{x}(t,0,y,\eta) - B'(t)\tilde{\xi}(t,0,y,\eta)| = 0.$$

3.1. Isotropic case. Consider the case where $\omega_1 = \cdots = \omega_d = \omega > 0$. Define $\theta_k \in C(\mathbf{R}^d \setminus \{0\}, \mathbf{R}^d)$ by

$$\theta_k(\eta) = \begin{cases} (2/\omega^2)\cdot n(F(\eta) - F(-\eta)) & \text{if } k = 2n,\ n \in \mathbf{Z}; \\ (2/\omega^2)\cdot \big(n(F(\eta) - F(-\eta)) + F(\eta)\big) & \text{if } k = 2n+1,\ n \in \mathbf{Z}. \end{cases}$$

As a corollary of Proposition 7, we have

Corollary 8. *For every compact set $K \subset \mathbf{R}^d$ and every $k \in \mathbf{Z}$,*

$$x(k\pi/\omega, y, \eta) = (-1)^k(y + \theta_k(\eta)) + o(1) \text{ as } |\eta| \to \infty \text{ uniformly in } y \in K;$$

$$\xi(k\pi/\omega, y, \eta) = (-1)^k\eta + O(1) \text{ as } |\eta| \to \infty \text{ uniformly in } y \in K.$$

We recall the the definition of H^s wave front set.

Definition 9. Let U be an open set in $\mathbf{R}^d$, and $f \in \mathcal{D}'(U)$. For $(x_0, \xi_0) \in U \times (\mathbf{R}^d \setminus \{0\}) \cong T^*U \setminus 0$, we say that $f \in H^s$ at (x_0, ξ_0) if there are a function $a \in C_0^\infty(U)$ satisfying $a = 1$ in a neighborhood of x_0, and a conic neighborhood Γ of ξ_0 in $\mathbf{R}^d \setminus \{0\}$ such that $\chi_\Gamma(D)(af) = \mathcal{F}^{-1}[\chi_\Gamma(\cdot)\mathcal{F}(af)(\cdot)] \in H^s$. Here $\chi_\Gamma(\xi) = 1$ if $\xi \in \Gamma$, and $\chi_\Gamma(\xi) = 0$ otherwise; and $\mathcal{F}$ denotes the Fourier transformation. The H^s wave front set of f, $WF_{H^s}f$, is the set of all $(x_0, \xi_0) \in T^*U \setminus 0$ such that $f \notin H^s$ at (x_0, ξ_0). $WF_{H^\infty} = WF$ is the usual wave front set.

Theorem 10. *Assume (W3) and (W4). Let $u_0 \in \mathcal{S}'(\mathbf{R}^d)$ and $u(t) = e^{-itH}u_0 \in C(\mathbf{R}, \mathcal{S}'(\mathbf{R}^d))$. Then for every $s \in \mathbf{R}$ and $k \in \mathbf{Z}$,*

$$WF_{H^s}u(k\pi/\omega) = \left\{(-1)^k(y + \theta_k(\eta),\ \eta);\ (y,\eta) \in WF_{H^s}u_0\right\}.$$

In particular, if $\lim_{|x|\to\infty} |\nabla W(x)| = 0$ in addition, then for every $s \in \mathbf{R}$ and $k \in \mathbf{Z}$,

$$WF_{H^s}u(k\pi/\omega) = \{(-1)^k(y,\eta); (y,\eta) \in WF_{H^s}u_0\}.$$

3.2. Anisotropic case. Consider the case where $\omega_j/\omega_k \in \mathbf{Q}$ for every j, k. Then we can write

$$\frac{\pi}{\omega_j} = \frac{T}{m_j}, \ m_j \in \mathbf{N} \ (j = 1, \ldots, d), \quad \text{g.c.d.}\{m_j; 1 \leq j \leq d\} = 1.$$

Here, for a subset S of $\mathbf{N}$, we denote by g.c.d. S the greatest common divisor of S. Then $A(kT) = \operatorname{diag}((-1)^{km_1}, \ldots, (-1)^{km_d})$ $(k \in \mathbf{Z})$. As a corollary of Proposition 7, we have

Corollary 11. *For every compact set $K \subset \mathbf{R}^d$ and every $k \in \mathbf{Z}$,*

$$x(kT, y, \eta) = A(kT)\tilde{x}(kT, 0, y, \eta) + o(1) \ \textit{as} \ |\eta| \to \infty \ \textit{uniformly in} \ y \in K;$$
$$\xi(kT, y, \eta) = A(kT)\eta + O(1) \ \textit{as} \ |\eta| \to \infty \ \textit{uniformly in} \ y \in K.$$

For $k \in \mathbf{Z}$, define a homeomorphism $\chi_k : T^*\mathbf{R}^d \setminus 0 \to T^*\mathbf{R}^d \setminus 0$ by

$$\chi_k(y, \eta) = e^{kTH_{h_0}}(\tilde{x}(kT, 0, y, \eta), \eta) = (A(kT)\tilde{x}(kT, 0, y, \eta), A(kT)\eta).$$

Then we have $\chi_{j+k} = \chi_j \circ \chi_k$ for every $j, k \in \mathbf{Z}$. This discrete flow describes the propagation of strong singularities for $t \in T\mathbf{Z}$.

Theorem 12. *Assume (W3) and (W4). Let $u_0 \in \mathcal{S}'(\mathbf{R}^d)$ and $u(t) = e^{-itH}u_0 \in C(\mathbf{R}, \mathcal{S}'(\mathbf{R}^d))$. Assume $\langle x \rangle^{-N} u_0 \in H^{s_0}$ for some $s_0 \in \mathbf{R}$ and $N \gg 1$. Then for every $s_0 < s \leq s_0 + 1$,*

$$WF_{H^s} u(kT) = \chi_k(WF_{H^s} u_0), \quad k \in \mathbf{Z}.$$

In particular, if $\lim_{|x| \to \infty} |\nabla W(x)| = 0$ in addition, then for every $s_0 < s \leq s_0 + 1$,

$$WF_{H^s} u(kT) = e^{kTH_{h_0}}(WF_{H^s} u_0), \quad k \in \mathbf{Z}.$$

The restriction on s can be removed if the singularities in the special directions determined from Q are concerned.

Theorem 13. *Assume (W3) and (W4). Let $u_0 \in \mathcal{S}'(\mathbf{R}^d)$ and $u(t) = e^{-itH}u_0 \in C(\mathbf{R}, \mathcal{S}'(\mathbf{R}^d))$. Let $\hat{\eta} \in \mathbf{R}^d \setminus \{0\}$ and set $I_{\hat{\eta}} = \{j \in \{1, 2, \ldots, d\}; \hat{\eta}_j \neq 0\}$. Assume that* g.c.d. $\{m_j; j \in I_{\hat{\eta}}\} = 1$. *Then for every $s \in \mathbf{R}$ and $k \in \mathbf{Z}$, the following two conditions on $y \in \mathbf{R}^d$ are equivalent:*

(i) $\chi_k(y, \hat{\eta}) \in WF_{H^s} u(kT)$;

(ii) $(y, \hat{\eta}) \in WF_{H^s} u_0$.

Remark. (1) If $\hat{\eta}_j \neq 0$ for all $j = 1, \dots, d$, then g.c.d. $\{m_j; j \in I_{\hat{\eta}}\} = 1$.

(2) In the isotropic case, we have $m_1 = \cdots = m_d = 1$; hence g.c.d. $\{m_j; j \in I_{\hat{\eta}}\} = 1$.

3.3. Creation of weaker singularities. We preserve the assumption and notation in Section 3.2 and consider the case where $W \in C_0^\infty(\mathbf{R}^d)$. If $u_0 \in H^{s_0}$, then Theorem 12 implies that

$$WF_{H^s}u(kT) = e^{kTH_{h_0}} WF_{H^s}u_0$$

for every $s_0 < s \leq s_0 + 1$ and $k \in \mathbf{Z}$. For $s > s_0 + 1$, $WF_{H^s}u(kT)$ $(k \in \mathbf{Z})$ are not stable in general; there exist some cases where new singularities of order $s > s_0 + 1$ can appear along a hypersurface even if $W \in C_0^\infty(\mathbf{R}^d)$.

Proposition 14. *Let* $\hat{\eta} = (1, 0, \dots, 0)$ *or* $(-1, 0, \dots, 0)$. *Assume that* $m_1 \neq 1$ *and that* g.c.d. $\{m_1, m_j\} = 1$ *for every* $j = 2, \dots, d$. *Let* $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{d-1}$, *and let* $u_0(x) = \phi_1(x_1)\phi_2(x')$, *where* $\phi_1 \in H^{s_0}(\mathbf{R})$, $\phi_1 \notin H^{s_0+\varepsilon}(\mathbf{R})$ *for every* $\varepsilon > 0$, *and* $\phi_2 \in C_0^\infty(\mathbf{R}^{d-1})$, *not identically zero. Let* $u(t) = e^{-itH}u_0 \in C(\mathbf{R}, \mathcal{S}'(\mathbf{R}^d))$. *Assume that*

$$\int_{-\infty}^{\infty} V(x_1, \cdot)dx_1 \in C_0^\infty(\mathbf{R}^{d-1})$$

is nonnegative (or nonpositive) and not identically zero. Then for every $s_0+1 < s \leq s_0+2$ *and* $k \in \mathbf{Z} \setminus \{0\}$, *the following two conditions on* $y \in \mathbf{R}^d$ *are equivalent:*

(i) $e^{kTH_{h_0}}(y, \hat{\eta}) \in WF_{H^s}u(kT)$;

(ii) $(y, \hat{\eta}) \in WF_{H^s}u_0 \cup \left(WF_{H^{s-1}}\phi_1 \times (\mathbf{R}^{d-1} \times \{0\})\right) \subset T^*\mathbf{R} \times T^*\mathbf{R}^{d-1}$.

Here the identification $T^*\mathbf{R}^d \ni (x, \xi) \longleftrightarrow ((x_1, \xi_1), (x', \xi')) \in T^*\mathbf{R} \times T^*\mathbf{R}^{d-1}$ *is used.*

Remark. In the proposition above, $WF_{H^s}u_0 = WF_{H^s}\phi_1 \times (\operatorname{supp}\phi_2 \times \{0\})$. So weaker singularities in $WF_{H^{s-1}}\phi_1 \times (\mathbf{R}^{d-1} \times \{0\})$ are created when $s_0 + 1 < s \leq s_0 + 2$. □

Remark. If $m_1 = 1$, then g.c.d. $\{m_j; j \in I_{\hat{\eta}}\} = m_1 = 1$. Hence the conditions on $y \in \mathbf{R}^d$, (i) and (ii), are equivalent for every $s \in \mathbf{R}$ and $k \in \mathbf{Z}$ by Theorem 13. □

References

[1] W. Craig, T. Kappeler and W. Strauss, Microlocal dispersive smoothing for the Schrödinger equation, Comm. Pure Appl. Math. **48** (1995), 769-860.

[2] S. Doi, Smoothness of solutions for Schrödinger equations with unbounded potentials, to appear in Publ. RIMS.

[3] S. Doi, Singularities of solutions of Schrödinger equations for perturbed harmonic oscillators. *Hyperbolic Problems and Related Topics*, 185–199, Internatinal Press, Somerville, MA, 2003.

[4] S. Doi, Dispersion of singularities of solutions for Schrödinger equations, to appear in Comm. Math. Phys.

[5] S. Doi, Propagation and creation of singularities of solutions for Schrödinger equations, in preparation.

[6] D. Fujiwara, Remarks on the convergence of the Feynman path integrals, Duke Math. J. **47** (1980), 559-600.

[7] L. Kapitanski and I. Rodianski, Regulated smoothing for Schrödinger evolution, Internat. Math. Res. Notices **2** (1996), 41-54.

[8] L. Kapitanski, I. Rodianski, and K. Yajima, On the fundamental solution of a perturbed harmonic oscillator, Topol. Methods in Nonlinear Anal. **9** (1997), 77-106.

[9] T. Ōkaji, Propagation of wave packets and smoothing properties of soluitons to Schrödinger equations with unbounded potential, preprint (version 8.4), 2000.

[10] A. Weinstein, A symbol class for some Schrödinger equations on $\mathbf{R}^n$, Amer. J. Math. **107** (1985), 1-21.

[11] J. Wunsch, The trace of the generalized harmonic oscillator, Ann. Inst. Fourier, Grenoble **49** (1999), 351-373.

[12] K. Yajima, Smoothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations, Comm. Math. Phys. **181** (1996), 605-629.

[13] K. Yajima, On fundamental solution of time dependent Schrödinger equations, Contemp. Math. **217** (1998), 49-68.

[14] S. Zelditch, Reconstruction of singularities for solutions of Schrödinger's equation, Comm. Math. Phys. **90** (1983), 1-26.